AD-A102 168 STANFORD UNIV CA DEPT OF STATISTICS F/G 12/1
LONG UNIMODAL SUBSEQUENCES: A PROBLEM OF F.R.K. CHUNG.(U)
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UNCLASSIFIED TR-301 NL

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6 LONG UNIMODAL SUBSEQUENCES: A PROBLEM OF F.R.K. CHUNG

By

10 J. Michael Steele

TECHNICAL REPORT NO. 301

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Prepared Under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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I. Introduction.

Let p denote a permutation of $\{1,2,\ldots,n\}$ and call $\{a_1 < a_2 < \ldots < a_t\}$ a unimodal subsequence provided there is a j such that

$$p(a_1) < p(a_2) < \ldots < p(a_j) > p(a_{j+1}) > \ldots > p(a_t)$$

or

$$p(a_1) > p(a_2) > \ldots > p(a_j) < p(a_{j+1}) < \ldots < p(a_t) .$$

Let $\ell(n)$ denote the expected length of the longest unimodal subsequence of a randomly permuted subsequence i.e. $\ell(n) = \sum_p \rho(p)/n!$, where $\rho(p)$ denotes the length of the longest unimodal subsequence of the permutation p.

F.R.K. Chung [1] conjectured that

$$\lim_{n \to \infty} \ell(n)/\sqrt{n} = C \text{ exists} .$$

The point of this note is to prove Chung's conjecture and show $C = 2\sqrt{2}$. Actually, Chung's conjecture is slightly more general than this introductory version, and this more general conjecture is obtained by the same proof.

II. Proof of F.R.K. Chung's Conjecture.

Suppose (X_i, Y_i), $1 \le i < \infty$ are independent and uniformly distributed in $[0,1]^2$. For any $A \subset [0,1]$ let

$$
\begin{aligned}
I_n(A) = \max\{k:\ & Y_{i_1} < Y_{i_2} < \dots < Y_{i_k} \text{ with} \\
& X_{i_1} < X_{i_2} < \dots < X_{i_k},\ X_{i_j} \in A \text{ and} \\
& i_j \in [1,\dots,n]\}
\end{aligned}
$$

and

$$
\begin{aligned}
D_n(A) = \max\{k:\ & Y_{i_1} > Y_{i_2} > \dots > Y_{i_k} \text{ with} \\
& X_{i_1} < X_{i_2} < \dots < X_{i_k},\ X_{i_j} \in A \text{ and} \\
& i_j \in [1,2,\dots,n]\}.
\end{aligned}
$$

Next set

$$
U_n = \max_{0 \le t \le 1} \{\max(I_n([0,t]) + D_n([t,1]),\ D_n([0,t]) + I_n([t,1]))\} .
$$

The desired proof will be obtained by applying known results to the random variable U_n. To begin it is easy to check that

$$
EU_n = \ell(n) .
$$

Next we note that by the work of Hammersley [2] and Kesten [3] that almost surely and in L^1 we have the limits

$$\text{(2.2)} \qquad \lim_{n\to\infty} I_n(A)/\sqrt{n} = C\sqrt{\lambda(A)} \quad \text{and} \quad \lim_{n\to\infty} D_n(A)/\sqrt{n} = C\sqrt{\lambda(A)}$$

where $\lambda(A)$ is the Lebesgue measure of $A \subset [0,1]$, and C is a universal constant. The work of Logan and Shepp [9] and Vershik and Kerov [5] established that $C = 2$.

For any N and $1 \leq k \leq N$ we define

$$U_n^N(k) = \max[I_n(0,k/n) + D_n((k-1)/N,1),\ D_n(0,k/N) + I_n((k-1))/N,1)]$$

and

$$U_n^N = \max_{1\leq k\leq N} U_n^N(k) \ .$$

Clearly, for all N, $U_n \leq U_n^N$ and by the above mentioned limit results we have

$$\lim_{n\to\infty} U_n^N/\sqrt{n} = 2 \max_{1\leq k\leq N} (\sqrt{k/N} + \sqrt{(N-k+1)/N}) \ ,$$

where the limit is almost sure and in L^1. The arbitrariness of N then shows $\limsup_{n\to\infty} U_n/\sqrt{n} \leq 2 \max_{0<t<1} (\sqrt{t} + \sqrt{1-t}) = 2\sqrt{2}$ a.s., so by Fatou's lemma we get $\limsup_{n\to\infty} \ell(n)/\sqrt{n} \leq 2\sqrt{2}$.

For the opposite direction note the trivial bound

$$U_n \geq I_n([0,\tfrac{1}{2}]) + D_n([\tfrac{1}{2},1])$$

so

$$\liminf_{n\to\infty} \ell(n)/\sqrt{n} \geq \liminf_{n\to\infty} E(I_n[0,\tfrac{1}{2}] + D_n[\tfrac{1}{2},1]) = 2\sqrt{2}$$

which completes the proof.

III. The Generalization.

Instead of allowing the subsequence to make "one turn" as in the unimodal case, one can consider subsequences which make k turns. Explicitly, let $\ell_k(n)$ be the expected length of the longest subsequence S of a random permutation with the following property:

S can be decomposed into $k+1$ segments which are monotone and which alternate between increasing and decreasing.

The method of the preceeding section can be used easily to show

$$\lim_{n \to \infty} \ell_k(n)/\sqrt{n} = 2\sqrt{k+1} \; ;$$

all one has to do is define the proper analogue $U_n(k)$ of U_n and argue as before. One should also note that the preceding bounds also prove the almost sure and L^1 convergence of $U_n(k)/\sqrt{n}$ to $2\sqrt{k+1}$.

References

[1] Chung, F.R.K., On Unimodal Subsequences, Bell Laboratories Technical Report (1979).

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[3] Kesten, H. Comments on J.F.C. Kingman's Subadditive Ergodic Theory, Ann. Prob. 1, (1973), 883-889.

[4] Logan, B.F. and Shepp, L.A., A Variational Problem for Young Tableau, Adv. in Math., 26, (1977), 206-222.

[5] Vershik, A.M. and Kerov, C.V., Asymptotics of the Plancherel Measure of the Symmetric Group and a Limiting Form for Young Tableau, Dokl. Akad. Nauk U.S.S.R., 233, (1977), 1024-1027.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
301

2. GOVT ACCESSION NO.
AD-A102168

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
LONG UNIMODAL SUBSEQUENCES: A PROBLEM OF F.R.K. CHUNG

5. TYPE OF REPORT & PERIOD COVERED
TECHNICAL REPORT

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
J. MICHAEL STEELE

8. CONTRACT OR GRANT NUMBER(s)
N00014-76-C-0475

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics
Stanford University
Stanford, CA 94305

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NR-042-267

11. CONTROLLING OFFICE NAME AND ADDRESS
Office Of Naval Research
Statistics & Probability Program Code 436
Arlington, VA 22217

12. REPORT DATE
APRIL 2, 1981

13. NUMBER OF PAGES
5

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Unimodal subsequence, monotone subsequence, random permutation, uniformly distributed random variables.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\ell(n)$ be the expected length of the longest unimodal subsequence of a random permutation. It is proved here that $\ell(n)/\sqrt{n}$ converges to $2\sqrt{2}$. This settles a conjecture of F.R.K. Chung.
sq root of n   sq. root of 2

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S/N 0102-LF-014-6601

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